Solution by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Let any given orthogonal system of circles be inverted with respect to a circle whose center is an intersection of two circles, one from each family, and neither of them a real circle. The result is a new orthogonal system containing two straight lines derived from the two circles, and each line is the locus of centers of the 'opposite' family of circles. Using these lines as axes of coördinates, the two circle families are $(x-a)^2+y^2=c^2$, $x^2+(y-b)^2=d^2$, in which, because the circles are orthogonal, $a^2+b^2=c^2+d^2$. Writing $c^2=a^2-k^2$ in the last equation gives $d^2=b^2+k^2$, and the circle families become $x^2+y^2-2ax+k^2=0$, $x^2+y^2-2by-k^2=0$.

The constants are now independent, but since any circle of one family is orthogonal to all of the other family it follows that a and b are the respective parameters. If now a and b are replaced by $k\coth 2v$ and $-k\cot 2u$, respectively, the system may be written $u+vi=\tan^{-1}\dfrac{x+yi}{k}$, which shows that it is isothermal. Thus the given system is also isothermal, since it may be obtained from this one by inversion. When k is 0 or ∞ the corresponding result is

$$u+vi=\frac{1}{x+yi},\text{ or } u+vi=x+yi.$$

274. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

If a straight line AB is placed between two intersecting straight lines MN and PQ and is made to revolve through all possible positions having A always in MN and B always in PQ, what is the locus of any point L in AB or AB produced?

I. Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

We can choose coördinate axes so that the equations to the given lines are $y=rx$, $z=a$; $y=-rx$, $z=-a$. Let the coördinates of A, B, L be, respectively, (h, rh, a), $(k, -rk, -a)$, (x, y, z). Then

$$\frac{x-h}{x-k}=\frac{y-rh}{y+rk}=\frac{z-a}{z+a}=\frac{AL}{BL}=m,\text{ say.}$$

$\therefore h-mk=x(1-m)$..............(1), $r(h+mk)=y(1-m)$...............(2), $z(1-m)=a(1+m)$................(3).

Hence the locus lies in a plane parallel to $z=0$, or to the given lines as is otherwise evident. Also $AB^2=l^2=(h-k)^2+r^2(h+k)^2+4a^2$................(4). Eliminating h, k between (1), (2), (4) we have an ellipse for the required locus, its equation being

$$(1-m)^2\{[y(m-1)+rx(m+1)]^2+[y(m+1)+rx(m-1)]^2r^2\}=4m^2r^2(l^2-4a^2).$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O be the intersection of MN, PQ. $OA=a$, $OB=b$. Draw LD parallel to OB, and let D be in MN. Let $OD=u$, $DL=v$, $AB=c$, $AL=d$, $\angle AOB=\beta$. Then $a^2+b^2-2ab\cos\beta=c^2$; $c:d=a:a\pm u$; $c:d=b:v$.

$$\text{Hence } a=\pm\frac{cu}{d-c},\ b=\frac{cv}{d},\text{ and }\frac{c^2u^2}{(d-c)^2}+\frac{c^2v^2}{d^2}\mp\frac{2c^2uv\cos\beta}{d(d-c)}=c^2.$$

$$\therefore\ \frac{u^2}{(d-c)^2}+\frac{v^2}{d^2}\mp\frac{2uv\cos\beta}{d(d-c)}=1.$$

$\therefore$ The locus is an ellipse.

III. Solution by A. H. HOLMES, Brunswick, Maine.

Suppose the straight lines MN and PQ intersect each other at right angles at O, and AB placed between them: A on MN and B on PQ, and L a point in AB. Draw LO. Put $AL=b$, $BL=a$, and $LO=r$, and $LAO=\phi$, $AOL=\theta$. Then $b\sin\phi=r\sin\theta$, and $a\cos\phi=r\cos\theta$.

$$\therefore \sin^2\phi=\frac{r^2}{b^2}\sin^2\theta,\text{ and }\cos^2\phi=\frac{r^2}{a^2}\cos^2\theta.\quad \therefore r=\frac{ab}{\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}}.$$

Therefore the locus of point L is an ellipse whose semi-major axis is BL and whose semi-minor axis is AL. When MN and PQ intersect obliquely at angle ψ the semi-minor axis would be $\dfrac{ab\sin\psi}{\sqrt{(a^2-b^2\cos^2\psi)}}$.

Also solved by R. D. Carmichael, and J. Scheffer.

275. Proposed by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

An hyperbola is drawn touching the axes of an ellipse, and the asymptotes of the hyperbola touch the ellipse. Prove that the center of the hyperbola lies on one of the equal conjugate diameters of the ellipse.

Solution by the PROPOSER.

Let (x', y') be the intersection of the tangents to the ellipse $a^2y^2+b^2x^2-a^2b^2=0$ .... (1); then these tangents being the asymptotes of the hyperbola, (x', y') is the center of the hyperbola. The equation to the tangents to (1) from (x', y') is

$$(a^2y^2+b^2x^2-a^2b^2)(a^2y'^2+b^2x'^2-a^2b^2)=(a^2y'y+b^2x'x-a^2b^2)^2 \text{..........(2)},$$

$$\text{or, } (y'^2-b^2)x^2+(x'^2-a^2)y^2-2x'y'xy+2b^2x'x+2a^2y'y-(a^2y'^2+b^2x'^2)=0 \text{.....(3)}.$$

Now, the equation to the asymptotes of a conic differs from the equation to the conic by a constant only; then adding c to the left member of (3) we have the equation to the hyperbola.

If now $y=0$ in this equation to the hyperbola, we have